17 Planar Straight Line Graphs

Siu-Wing Cheng
Hong Kong University of Science and Technology

17.1 Introduction

Graphs (Chapter 4) have found extensive applications in computer science as a modeling tool. In mathematical terms, a graph is simply a collection of vertices and edges. Indeed, a popular graph data structure is the adjacency lists representation [14] in which each vertex keeps a list of vertices connected to it by edges. In a typical application, the vertices model entities and an edge models a relation between the entities corresponding to the edge endpoints. For example, the transportation problem calls for a minimum cost shipping pattern from a set of origins to a set of destinations [2]. This can be modeled as a complete directed bipartite graph. The origins and destinations are represented by two columns of vertices. Each origin vertex is labeled with the amount of supply stored there. Each destination vertex is labeled with the amount of demand required there. The edges are directed from the origin vertices to the destination vertices and each edge is labeled with the unit cost of transportation. Only the adjacency information between vertices and edges are useful and captured, apart from the application dependent information.

In geometric computing, graphs are also useful for representing various diagrams. We restrict our attention to diagrams that are planar graphs embedded in the plane using straight edges without edge crossings. Such diagrams are called *planar straight line graphs* and denoted by PSLGs for short. Examples include Voronoi diagrams, arrangements, and triangulations. Their definitions can be found in standard computational geometry texts such as the book by de Berg et al. [3]. See also Chapters 62, 63 and 64. For completeness, we also provide their definitions in section 17.8. The straight edges in a PSLG partition the plane into regions with disjoint interior. We call these regions *faces*. The adjacency lists representation is usually inadequate for applications that manipulate PSLGs. Consider the problem of locating the face containing a query point in a Delaunay triangulation. One practical algorithm is to walk towards the query point from a randomly chosen starting vertex [11], see Figure 17.1. To support this algorithm, one needs to know the first face

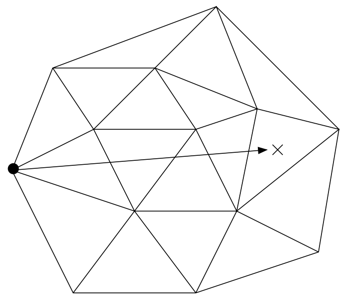


FIGURE 17.1: Locate the face containing the cross by walking from a randomly chosen vertex.

that we enter as well as the next face that we step into whenever we cross an edge. Such information is not readily provided by an adjacency lists representation.

There are three well-known data structures for representing PSLGs: the winged-edge, halfedge, and quadedge data structures. In Sections 17.2 and 17.3, we discuss the PSLGs that we deal with in more details and the operations on PSLGs. Afterwards, we introduce the three data structures in Section 17.4–17.6. We conclude in Section 17.7 with some further remarks.

17.2 Features of PSLGs

We assume that each face has exactly one boundary and we allow dangling edges on a face boundary. These assumptions are valid for many important classes of PSLGs such as triangulations, Voronoi diagrams, planar subdivisions with no holes, arrangements of lines, and some special arrangements of line segments (see Figure 17.2).

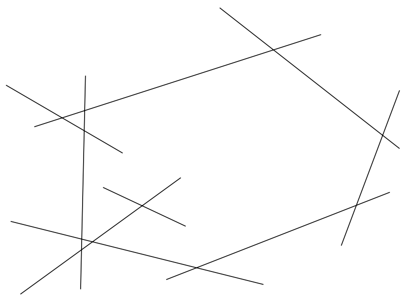


FIGURE 17.2: Dangling edges.

There is at least one unbounded face in a PSLG but there could be more than one, for example, in the arrangement of lines shown in Figure 17.3. The example also shows that there may be some infinite edges. To handle infinite edges like halflines and lines, we need a special vertex v_{inf} at infinity. One can imagine that the PSLG is placed in a small almost flat disk D at the north pole of a giant sphere S and v_{inf} is placed at the south pole. If an edge e is a halfline originating from a vertex u, then the endpoints of e are u and v_{inf}.

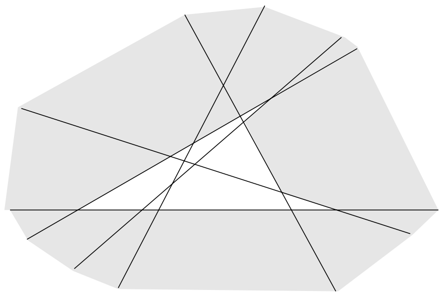


FIGURE 17.3: The shaded faces are the unbounded faces of the arrangement.

One can view e as a curve on S from u near the north pole to v_{inf} at the south pole, but e behaves as a halfline inside the disk D. If an edge e is a line, then v_{inf} is the only endpoint of e. One can view e as a loop from v_{inf} to the north pole and back, but e behaves as a line inside the disk D.

We do not allow isolated vertices, except for v_{inf}. Planarity implies that the incident edges of each vertex are circularly ordered around that vertex. This applies to v_{inf} as well.

A PSLG data structure keeps three kinds of attributes: *vertex attributes*, *edge attributes*, and *face attributes*. The attributes of a vertex include its coordinates except for v_{inf} (we assume that v_{inf} is tagged to distinguish it from other vertices). The attributes of an edge include the equation of the support line of the edge (in the form of $Ax + By + C = 0$). The face attributes are useful for auxiliary information, e.g., color.

17.3 Operations on PSLGs

The operations on a PSLG can be classified into *access functions* and *structural operations*. The access functions retrieve information without modifying the PSLG. Since the access functions partly depend on the data structure, we discuss them later when we introduce the data structures. In this section, we discuss four structural operations on PSLGs: *edge insertion*, *edge deletion*, *vertex split*, and *edge contraction*. We concentrate on the semantics of these four operations and discuss the implementation details later when we introduce the data structures. For vertex split and edge contraction, we assume further that each face in the PSLG is a simple polygon as these two operations are usually used under such circumstances.

Edge insertion and deletion

When a new edge e with endpoints u and v is inserted, we assume that e does not cross any existing edge. If u or v is not an existing vertex, the vertex will be created. If both u and v are new vertices, e is an isolated edge inside a face f. Since each face is assumed to have exactly one boundary, this case happens only when the PSLG is empty and f is the entire plane. Note that e becomes a new boundary of f. If either u or v is a new vertex, then the boundary of exactly one face gains the edge e. If both u and v already exist, then u and v lie on the boundary of a face which is split into two new faces by the insertion of e. These cases are illustrated in Figure 17.4.

The deletion of an edge e has the opposite effects. After the deletion of e, if any of its endpoint becomes an isolated vertex, it will be removed. The vertex v_{inf} is an exception and it is the only possible isolated vertex. The edge insertion is clearly needed to create

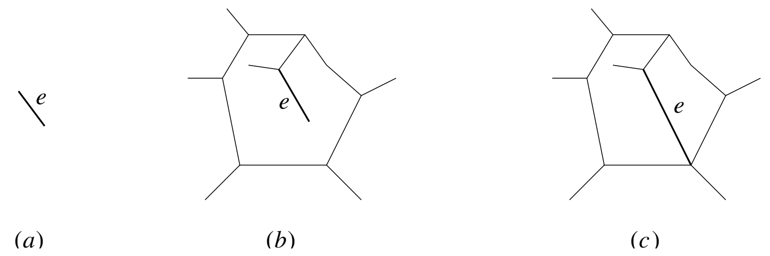


FIGURE 17.4: Cases in edge insertion.

a PSLG from scratch. Other effects can be achieved by combining edge insertions and deletions appropriately. For example, one can use the two operations to overlay two PSLGs in a plane sweep algorithm, see Figure 17.5.

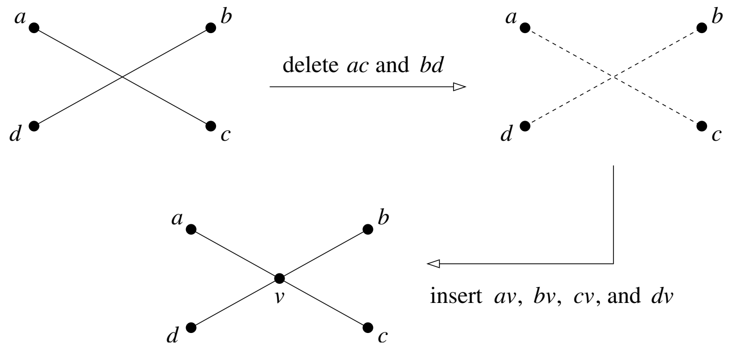


FIGURE 17.5: Intersecting two edges.

Vertex split and edge contraction

The splitting of a vertex v is best visualized as the continuous morphing of v into an edge e. Depending on the specification of the splitting, an incident face of v gains e on its boundary or an incident edge of v is split into a triangular face, see Figure 17.6. The incident edges of v are displaced and it is assumed that no self-intersection occurs within the PSLG during the splitting. The contraction of an edge e is the inverse of the vertex split. We also assume that no self-intersection occurs during the edge contraction. If e is incident on a triangular face, that face will disappear after the contraction of e.

Not every edge can be contracted. Consider an edge ab. If the PSLG contains a cycle abv that is not the boundary of any face incident to ab, we call the edge ab *non-contractible* because its contraction is not cleanly defined. Figure 17.7 shows an example. In the figure, after the contraction, there is an ambiguity whether dv should be incident on the face f_1 or the face f_2. In fact, one would expect the edge dv to behave like av and bv and be incident on both f_1 and f_2 after the contraction. However, this is impossible.

The vertex split and edge contraction have been used in clustering and hierarchical drawing of maximal planar graphs [6].

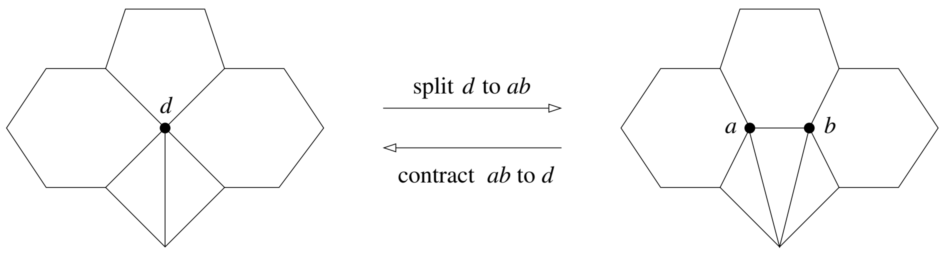


FIGURE 17.6: Vertex split and edge contraction.

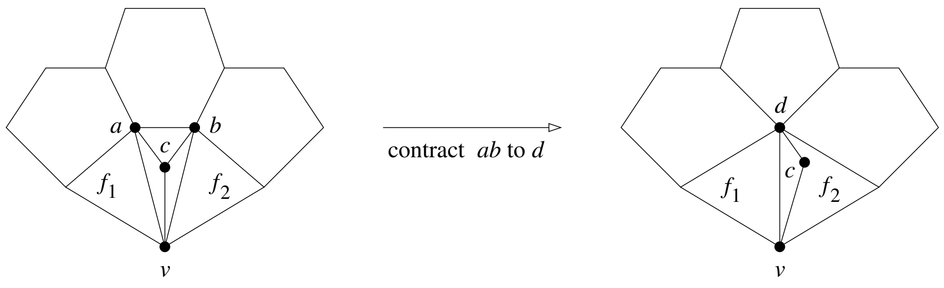


FIGURE 17.7: Non-contractible edge.

17.4 Winged-Edge

The winged-edge data structure was introduced by Baumgart [1] and it predates the halfedge and quadedge data structures. There are three kinds of records: *vertex records*, *edge records*, and *face records*. Each vertex record keeps a reference to one incident edge of the vertex. Each face record keeps a reference to one boundary edge of the face. Each edge e is stored as an oriented edge with the following references (see Figure 17.8):

- The *origin* endpoint $e.org$ and the *destination* endpoint $e.dest$ of e. The convention is that e is directed from $e.org$ to $e.dest$.
- The faces $e.left$ and $e.right$ on the left and right of e, respectively.
- The two edges $e.lcw$ and $e.lccw$ adjacent to e that bound the face $e.left$. The edge $e.lcw$ is incident to $e.org$ and the edge $e.lccw$ is incident to $e.dest$. Note that $e.lcw$ (resp. $e.lccw$) succeeds e if the boundary of $e.left$ is traversed in the clockwise (resp. anti-clockwise) direction from e.
- The two edges $e.rcw$ and $e.rccw$ adjacent to e that bound the face $e.right$. The edge $e.rcw$ is incident to $e.dest$ and the edge $e.rccw$ is incident to $e.org$. Note that $e.rcw$ (resp. $e.rccw$) succeeds e if the boundary of $e.right$ is traversed in the clockwise (resp. anti-clockwise) direction from e.

The information in each edge record can be retrieved in constant time. Given a vertex v, an edge e, and a face f, we can thus answer in constant time whether v is incident on e and e is incident on f. Given a vertex v, we can traverse the edges incident to v in clockwise order as follows. We output the edge e kept at the vertex record for v. We perform $e := e.rccw$ if $v = e.org$ and $e := e.lccw$ otherwise. Then we output e and repeat the above. Given a face f, we can traverse its boundary edges in clockwise order as follows. We output the edge e kept at the face record for f. We perform $e := e.lcw$ if $f = e.left$ and $e := e.rcw$ otherwise.

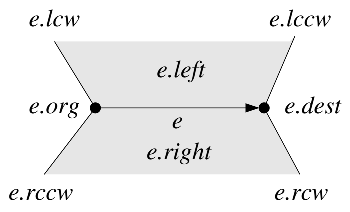


FIGURE 17.8: Winged-edge data structure.

Then we output e and repeat the above.

Note that an edge reference does not carry information about the orientation of the edge. Also, the orientations of the boundary edges of a face need not be consistent with either the clockwise or anti-clockwise traversal. Thus, the manipulation of the data structure is often complicated by case distinctions. We illustrate this with the insertion of an edge e. Assume that $e.org = u$, $e.dest = v$, and both u and v already exist. The input also specifies two edges e_1 and e_2 incident to u and v, respectively. The new edge e is supposed to immediately succeed e_1 (resp. e_2) in the anti-clockwise ordering of edges around u (resp. v). The insertion routine works as follows.

1. If $u = v_{\text{inf}}$ and it is isolated, we need to store the reference to e in the vertex record for u. We update the vertex record for v similarly.
2. Let e_3 be the incident edge of u following e_1 such that e is to be inserted between e_1 and e_3. Note that e_3 succeeds e_1 in anti-clockwise order. We insert e between e_1 and e_3 as follows.

```
$e.rccw := e_1$; $e.lcw := e_3$;
**if** $e.org = e_1.org$ **then** $e_1.lcw := e$; **else** $e_1.rcw := e$;
**if** $e.org = e_3.org$ **then** $e_3.rccw := e$; **else** $e_3.lccw := e$;
```

3. Let e_4 be the incident edge of v following e_2 such that e is to be inserted between e_2 and e_4. Note that e_4 succeeds e_2 in anti-clockwise order. We insert e between e_2 and e_4 as follows.

```
$e.lccw := e_2$; $e.rcw := e_4$;
**if** $e.dest = e_2.dest$ **then** $e_2.rcw := e$; **else** $e_2.lcw := e$;
**if** $e.dest = e_4.dest$ **then** $e_4.lccw := e$; **else** $e_4.rccw := e$;
```

4. The insertion of e has split a face into two. So we create a new face f and make *e.left* reference it. Also, we store a reference to e in the face record for f. There are further ramifications. First, we make *e.right* reference the old face.

```
**if** $e.org = e_1.org$ **then** $e.right := e_1.left$; **else** $e.right := e_1.right$;
```

Second, we make the *left* or *right* fields of the boundary edges of f reference f.

```
$e' := e$; $w := e.org$;
**repeat**
   **if** $e'.org = w$ **then** $e'.left := f$; $w := e'.dest$; $e' := e'.lccw$
                  **else** $e'.right := f$; $w := e'.org$; $e' := e'.rccw$
**until** $e' = e$;
```

Notice the inconvenient case distinctions needed in steps 2, 3, and 4. The halfedge data structure is designed to keep both orientations of the edges and link them properly. This eliminates most of these case distinctions as well as simplifies the storage scheme.

17.5 Halfedge

In the halfedge data structure, for each edge in the PSLG, there are two symmetric edge records for the two possible orientations of the edge [15]. This solves the orientation problem in the winged-edge data structure. The halfedge data structure is also known as the *doubly connected edge list* [3]. We remark that the name doubly connected edge list was first used to denote the variant of the winged-edge data structure in which the *lccw* and *rccw* fields are omitted [12, 13].

There are three kinds of records: *vertex records*, *halfedge records*, and *face records*. Let e be a halfedge. The following information is kept at the record for e (see Figure 17.9).

- The reference $e.sym$ to the symmetric version of e.
- The *origin* endpoint $e.org$ of e. We do not need to store the destination endpoint of e since it can be accessed as $e.sym.org$. The convention is that e is directed from $e.org$ to $e.sym.org$.
- The face $e.left$ on the left of e.
- The next edge $e.succ$ and the previous edge $e.pred$ in the anti-clockwise traversal around the face $e.left$.

For each vertex v, its record keeps a reference to one halfedge $v.edge$ such that $v = v.edge.org$. For each face f, its record keeps a reference to one halfedge $f.edge$ such that $f = f.edge.left$.

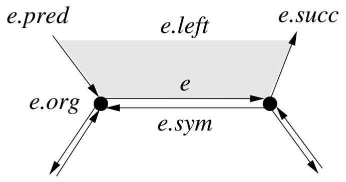


FIGURE 17.9: Halfedge data structure.

We introduce two basic operations `make_halfedges` and `half_splice` which will be needed for implementing the operations on PSLGs. These two operations are motivated by the operations `make_edge` and `splice` introduced by Guibas and Stolfi [8] for the quadedge data structure. We can also do without `make_halfedges` and `half_splice`, but they make things simpler.

- `make_halfedges`(u, v): Return two halfedges e and $e.sym$ connecting the points u and v. The halfedges e and $e.sym$ are initialized such that they represent a new PSLG with e and $e.sym$ as the only halfedges. That is, $e.succ = e.sym = e.pred$ and $e.sym.succ = e = e.sym.pred$. Also, e is the halfedge directed from u to v. If u and v are omitted, it means that the actual coordinates of $e.org$ and $e.sym.org$ are unimportant.
- `half_splice`(e_1, e_2): Given two halfedges e_1 and e_2, `half_splice` swaps the contents of $e_1.pred$ and $e_2.pred$ and the contents of $e_1.pred.succ$ and $e_2.pred.succ$. The effects are:
  - Let $v = e_2.org$. If $e_1.org \neq v$, the incident halfedges of $e_1.org$ and $e_2.org$ are merged into one circular list (see Figure 17.10(a)). The vertex v is now

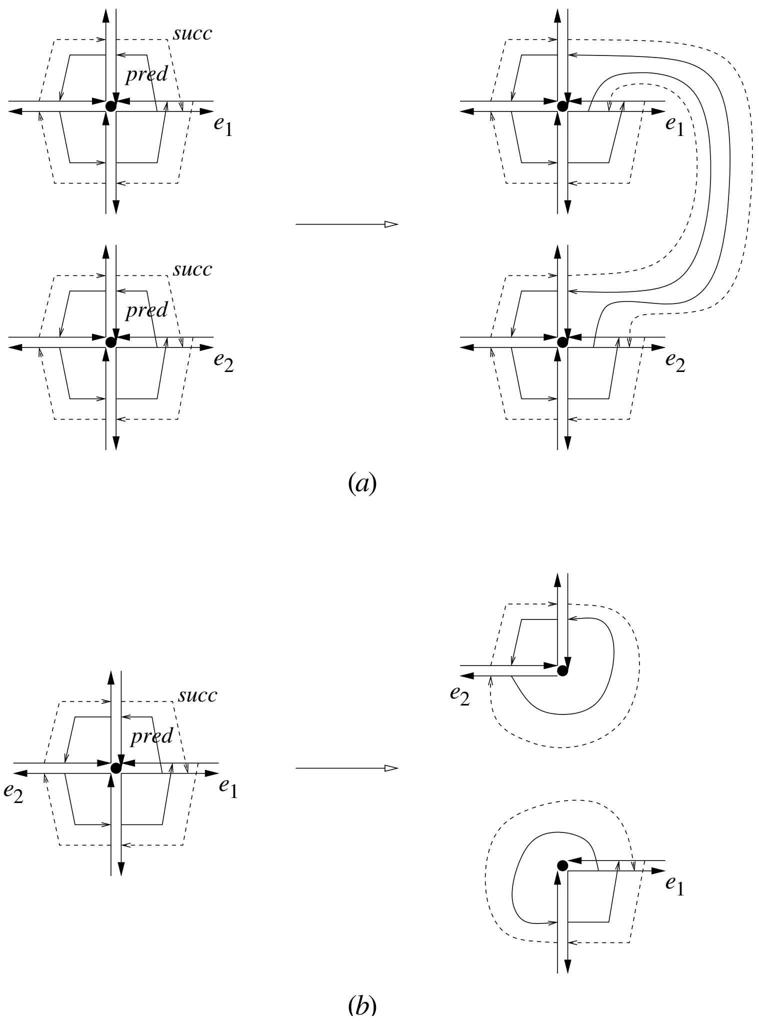


FIGURE 17.10: The effects of `half_splice`.

redundant and we finish the merging as follows.

```
$e' := e_2$;
repeat
    $e'.org := e_1.org$; $e' := e'.sym.succ$;
until $e' = e_2$;
delete the vertex record for $v$;
```

– Let $v = e_2.org$. If $e_1.org = v$, the incident halfedges of v are separated into two circular lists (see Figure 17.10(b)). We create a new vertex u for $e_2.org$ with the coordinates of u left uninitialized. Then we finish the separation as follows.

```
$u.edge := e_2$; $e' := e_2$;
repeat
    $e'.org := u$; $e' := e'.sym.succ$;
until $e' = e_2$.
```

The behavior of `half_splice` is somewhat complex even in the following special cases. If e is an isolated halfedge, `half_splice`(e_1, e) deletes the vertex record for $e.org$ and makes e a halfedge incident to $e_1.org$ following e_1 in anti-clockwise order. If $e_1 = e.sym.succ$, `half_splice`(e_1, e) detaches e from the vertex $e_1.org$ and creates a new vertex record for $e.org$. If $e_1 = e$, `half_splice`(e, e) has no effect at all.

Access functions

The information in each halfedge record can be retrieved in constant time. Given a vertex v, a halfedge e, and a face f, we can thus answer the following adjacency queries:

1: Is v incident on e? This is done by checking if $v = e.org$ or $e.sym.org$.
2: Is e incident on f? This is done by checking if $f = e.left$.
3: List the halfedges with origin v in clockwise order. Let $e = v.edge$. Output e, perform $e := e.sym.succ$, and then repeat until we return to $v.edge$.
4: List the boundary halfedges of f in anti-clockwise order. Let $e = f.edge$. Output e, perform $e := e.succ$, and then repeat until we return to $f.edge$.

Other adjacency queries (e.g., listing the boundary vertices of a face) can be answered similarly.

Edge insertion and deletion

The edge insertion routine takes two vertices u and v and two halfedges e_1 and e_2. If u is a new vertex, e_1 is ignored; otherwise, we assume that $e_1.org = u$. Similarly, if v is a new vertex, e_2 is ignored; otherwise, we assume that $e_2.org = v$. The general case is that an edge connecting u and v is inserted between e_1 and $e_1.pred.sym$ and between e_2 and $e_2.pred.sym$. The two new halfedges e and $e.sym$ are returned with the convention that e is directed from u to v.

```
Algorithm insert($u, v, e_1, e_2$)
1.  $(e, e.sym)$ := make_halfedges($u, v$);
2.  if $u$ is not new
3.     then half_splice($e_1, e$);
4.              $e.left := e_1.left$;
5.              $e.sym.left := e_1.left$;
6.  if $v$ is not new
7.     then half_splice($e_2, e.sym$);
8.              $e.left := e_2.left$;
9.              $e.sym.left := e_2.left$;
10. if neither $u$ nor $v$ is new
11.    then /* A face has been split */
12.             $e_2.left.edge := e$;
13.             create a new face $f$;
14.             $f.edge := e.sym$;
15.             $e' := e.sym$;
16.             repeat
17.                  $e'.left := f$;
18.                  $e' := e'.succ$;
19.             until $e' = e.sym$;
20. return $(e, e.sym)$;
```

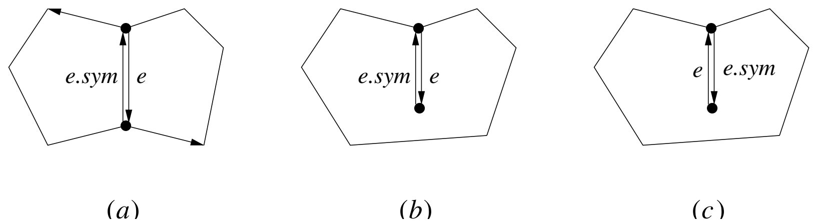


FIGURE 17.11: Cases in deletion.

The following deletion algorithm takes the two halfedges e and $e.sym$ corresponding to the edge to be deleted. If the edge to be deleted borders two adjacent faces, they have to be merged after the deletion.

```
Algorithm delete($e, e.sym$)
1.  if $e.left \neq e.sym.left$
2.     then /* Figure 17.11(a) */
3.            /* the faces adjacent to $e$ and $e.sym$ are to be merged */
4.            delete the face record for $e.sym.left$;
5.            $e' := e.sym$;
6.            repeat
7.                 $e'.left := e.left$;
8.                 $e' := e'.succ$;
9.            until $e' = e.sym$;
10.           $e.left.edge := e.succ$;
11.           half_splice($e.sym.succ, e$);
12.           half_splice($e.succ, e.sym$);
13.    else  if $e.succ = e.sym$
14.             then /* Figure 17.11(b) */
15.                    $e.left.edge := e.pred$;
16.                    half_splice($e.sym.succ, e$);
17.             else  /* Figure 17.11(c) */
18.                    $e.left.edge := e.succ$;
19.                    half_splice($e.succ, e.sym$);
20. /* $e$ becomes an isolated edge */
21. delete the vertex record for $e.org$ if $e.org \neq v_{\text{inf}}$;
22. delete the vertex record for $e.sym.org$ if $e.sym.org \neq v_{\text{inf}}$;
23. delete the halfedges $e$ and $e.sym$;
```

Vertex split and edge contraction

Recall that each face is assumed to be a simple polygon for the vertex split and edge contraction operations. The vertex split routine takes two points (p, q) and (x, y) and four halfedges e_1, e_2, e_3, and e_4 in anti-clockwise order around the common origin v. It is required that either $e_1 = e_2$ or $e_1.pred = e_2.sym$ and either $e_3 = e_4$ or $e_3.pred = e_4.sym$. The routine splits v into an edge e connecting the points (p, q) and (x, y). Also, e borders the faces bounded by e_1 and e_2 and by e_3 and e_4. Note that if $e_1 = e_2$, we create a new face bounded by e_1, e_2, and e. Similarly, a new face is created if $e_3 = e_4$. The following is the vertex split algorithm.

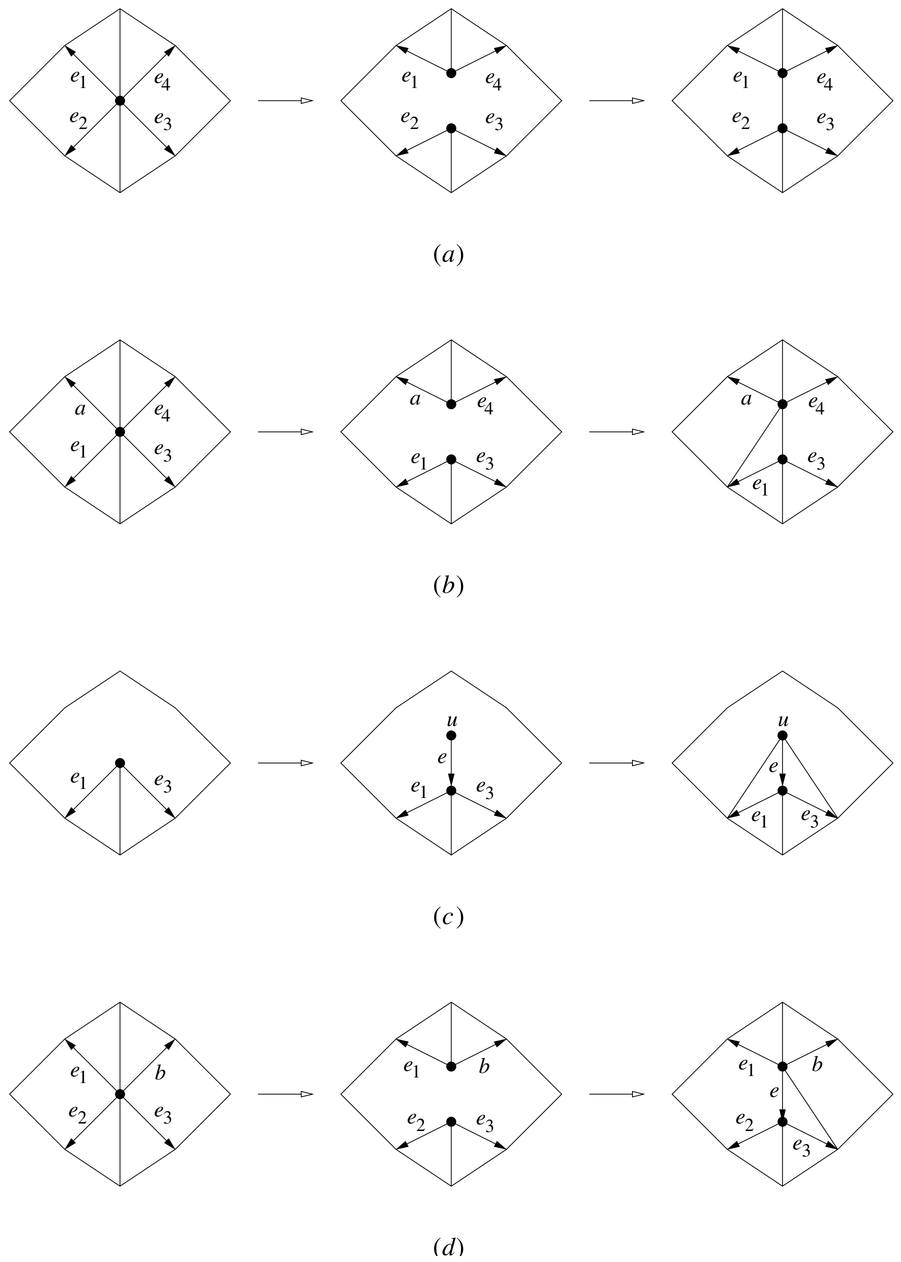


FIGURE 17.12: Cases for `split`.

```
Algorithm split($p, q, x, y, e_1, e_2, e_3, e_4$)
1.  if $e_1 \neq e_2$ and $e_3 \neq e_4$
2.      then /* Figure 17.12(a) */
3.          half_splice($e_1, e_3$);
4.          insert($e_1.org, e_3.org, e_1, e_3$);
5.          set the coordinates of $e_3.org$ to $(x, y)$;
6.          set the coordinates of $e_1.org$ to $(p, q)$;
```

```
7.    else  if $e_1 = e_2$
8.             then $a := e_1.sym.succ$;
9.                  if $a \neq e_3$
10.                   then /* Figure 17.12(b) */
11.                        half_splice($a, e_3$);
12.                        insert($a.org, e_3.org, a, e_3$);
13.                        insert($a.org, e_1.sym.org, a, e_1.sym$);
14.                        set the coordinates of $a.org$ to $(x, y)$;
15.                   else /* Figure 17.12(c) */
16.                        let $u$ be a new vertex at $(x, y)$;
17.                        $(e, e.sym)$ := insert($u, e_1.org, \cdot, e_3$);
18.                        insert($u, e_1.sym.org, e, e_1.sym$);
19.                        insert($u, e_3.sym.org, e, e_3.succ$);
20.                  set the coordinates of $e_1.org$ to $(p, q)$;
21.            else $b := e_3.pred.sym$;
22.                  /* since $e_1 \neq e_2$, $b \neq e_2$ */
23.                  /* Figure 17.12(d) */
24.                  half_splice($e_1, e_3$);
25.                  $(e, e.sym)$ := insert($b.org, e_3.org, e_1, e_3$);
26.                  insert($b.org, e_3.sym.org, e, e_3.succ$);
27.                  set the coordinates of $b.org$ to $(x, y)$;
28.                  set the coordinates of $e_3.org$ to $(p, q)$;
```

The following algorithm contracts an edge to a point (x, y), assuming that the edge contractibility has been checked.

```
Algorithm contract($e, e.sym, x, y$)
1.  $e_1 := e.succ$;
2.  $e_2 := e.pred.sym$;
3.  $e_3 := e.sym.succ$;
4.  $e_4 := e.sym.pred.sym$;
5.  delete($e, e.sym$);
6.  if $e_1.succ \neq e_2.sym$ and $e_3.succ \neq e_4.sym$
7.    then /* Figure 17.13(a) */
8.          half_splice($e_1, e_3$);
9.    else  if $e_1.succ = e_2.sym$ and $e_3.succ \neq e_4.sym$
10.             then /* Figure 17.13(b) */
11.                  delete($e_2, e_2.sym$);
12.                  half_splice($e_1, e_3$);
13.             else if $e_1.succ \neq e_2.sym$ and $e_3.succ = e_4.sym$
14.                   then /* symmetric to Figure 17.13(b) */
15.                        delete($e_4, e_4.sym$);
16.                        half_splice($e_1, e_3$);
17.                   else /* Figure 17.13(c) */
18.                        $a := e_3.sym.succ$;
19.                        delete($e_3, e_3.sym$);
20.                        if $a \neq e_2$
21.                          then delete($e_2, e_2.sym$);
22.                               half_splice($e_1, a$);
23.                          else delete($e_2, e_2.sym$);
24. set the coordinates of $e_1.org$ to $(x, y)$;
```

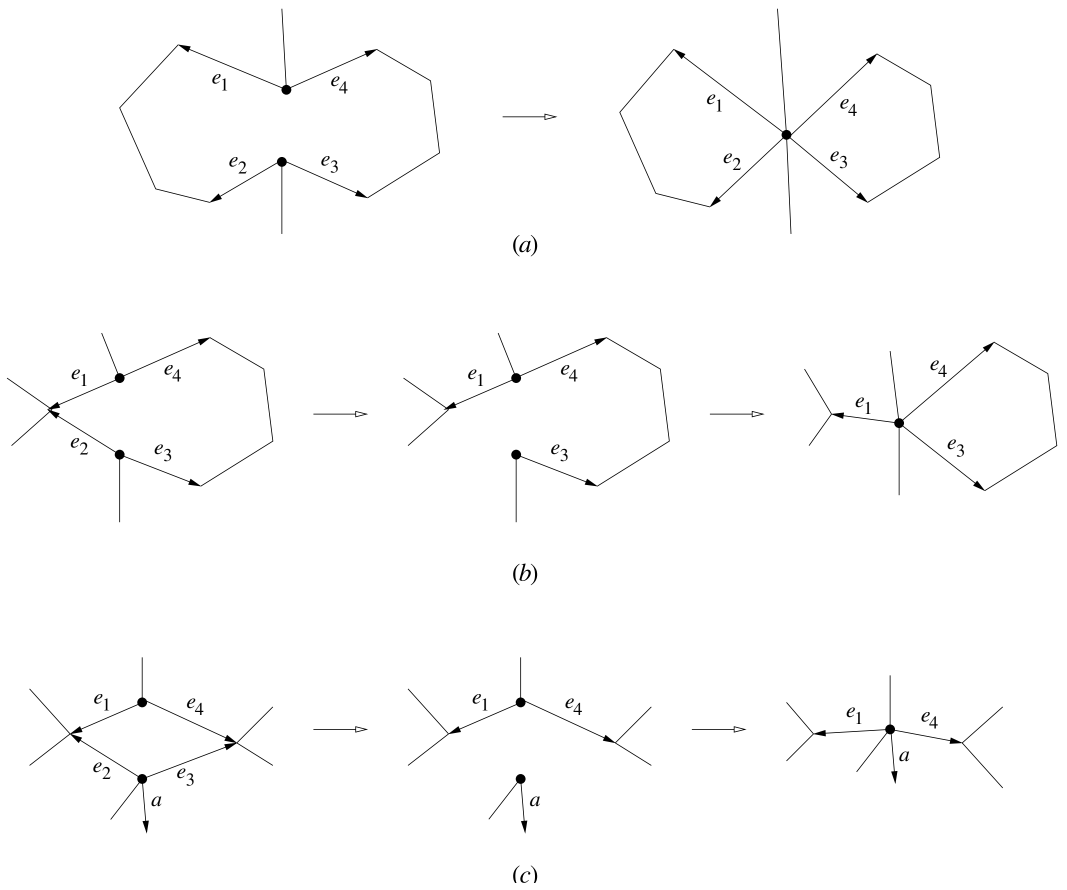


FIGURE 17.13: Cases for `contract`.

17.6 Quadedge

The quadedge data structure was introduced by Guibas and Stolfi [8]. It represents the planar subdivision and its dual simultaneously. The dual S^* of a PSLG S is constructed as follows. For each face of S, put a dual vertex inside the face. For each edge of S bordering the faces f and f', put a dual edge connecting the dual vertices of f and f'. The dual of a vertex v in S is a face and this face is bounded by the dual of the incident edges of v. Figure 17.14 shows an example. The dual may have loops and two vertices may be connected by more than one edge, so the dual may not be a PSLG. Nevertheless, the quadedge data structure is expressive enough to represent the dual. In fact, it is powerful enough to represent subdivisions of both orientable and non-orientable surfaces. We describe a simplified version sufficient for our purposes.

Each edge e in the PSLG is represented by four quadedges $e[i]$, where $i \in \{0, 1, 2, 3\}$. The quadedges $e[0]$ and $e[2]$ are the two oriented versions of e. The quadedges $e[1]$ and $e[3]$ are the two oriented versions of the dual of e. These four quadedges are best viewed as a cross such as $e[i+1]$ is obtained by rotating $e[i]$ for $\pi/2$ in the anti-clockwise direction. This is illustrated in Figure 17.15. The quadedge $e[i]$ has a *next* field referencing the quadedge that has the same origin as $e[i]$ and follows $e[i]$ in anti-clockwise order. In effect, the *next* fields form a circular linked list of quadedges with a common origin. This is called an *edge ring*.

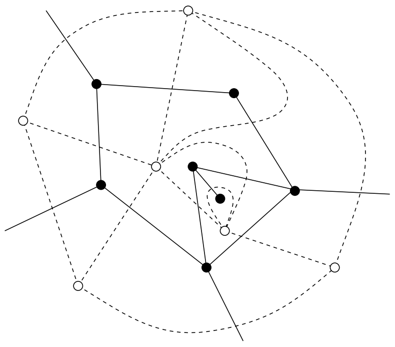


FIGURE 17.14: The solid lines and black dots show a PSLG and the dashed lines and the white dots denote the dual.

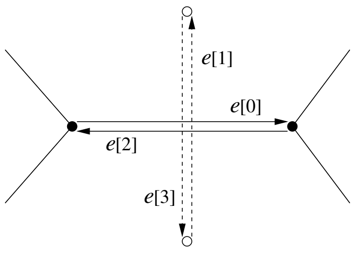


FIGURE 17.15: Quadedges.

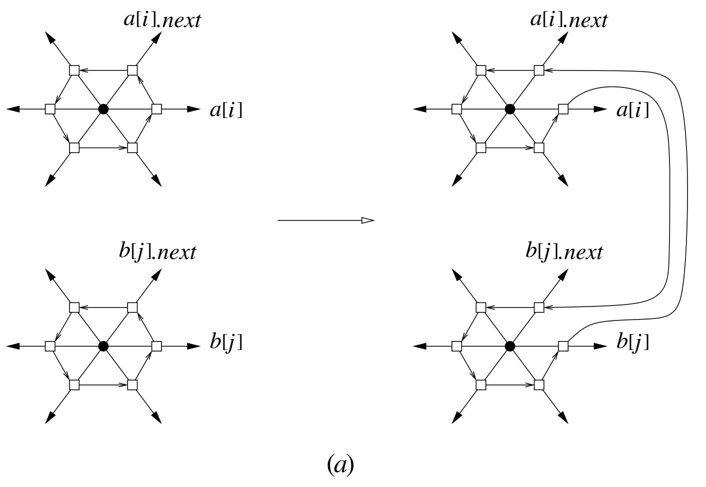
The following primitives are needed.

- `rot`(e, i): Return $e[(i+1) \bmod 4]$.
- `rot`$^{-1}(e, i)$: Return $e[(i+3) \bmod 4]$.
- `sym`(e, i): This function returns the quadedge with the opposite orientation of $e[i]$. This is done by returning `rot`(`rot`(e, i)).
- `onext`(e, i): Return $e[i].next$.
- `oprev`(e, i): This function gives the quadedge that has the same origin as $e[i]$ and follows $e[i]$ in clockwise order. This is done by returning `rot`$(e[(i{+}1) \bmod 4].next)$.

The quadedge data structure is entirely edge based and there are no explicit vertex and face records.

The following two basic operations `make_edge` and `splice` are central to the operations on PSLGs supported by the quadedge data structure. Our presentation is slightly different from that in the original paper [8].

- `make_edge`(u, v): Return an edge e connecting the points u and v. The quadedges $e[i]$ where $0 \leq i \leq 3$ are initialized such that they represent a new PSLG with e as the only edge. Also, $e[0]$ is the quadedge directed from u to v. If u and v are omitted, it means that the actual coordinates of the endpoints of are unimportant.
- `splice`(a, i, b, j): Given two quadedges $a[i]$ and $b[j]$, let $(c, k) =$ `rot`$(a[i].next)$ and $(d, l) =$ `rot`$(b[j].next)$, `splice` swaps the contents of $a[i].next$ and $b[j].next$ and the contents of $c[k].next$ and $d[l].next$. The effects on the edge rings of the origins of $a[i]$ and $b[j]$ and the edge rings of the origins of $c[k]$ and $d[l]$ are:



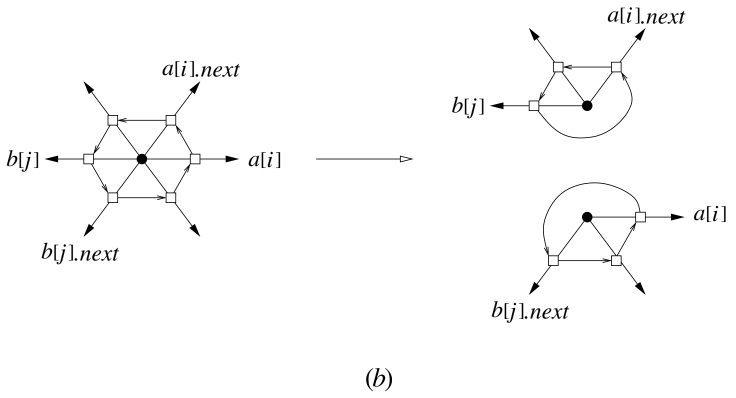


FIGURE 17.16: The effects of `splice`.

– If the two rings are different, they are merged into one (see Figure 17.16(a)).

– If the two rings are the same, it will be split into two separate rings (see Figure 17.16(b)).

Notice that `make_edge` and `splice` are similar to the operations `make_halfedges` and `half_splice` introduced for the halfedge data structure in the previous section. As mentioned before, they inspire the definitions of `make_halfedges` and `half_splice`. Due to this similarity, one can easily adapt the edge insertion, edge deletion, vertex split, and edge contraction algorithms in the previous section for the quadedge data structure.

17.7 Further Remarks

We have assumed that each face in the PSLG has exactly one boundary. This requirement can be relaxed for the winged-edge and the halfedge data structures. One method works as follows. For each face f, pick one edge from each boundary and keep a list of references to these edges at the face record for f. Also, the edge that belongs to outer boundary of f is specially tagged. With this modification, one can traverse the boundaries of a face

f consistently (e.g., keeping f on the left of traversal direction). The edge insertion and deletion algorithms also need to be enhanced. Since a face f may have several boundaries, inserting an edge may combine two boundaries without splitting f. If the insertion indeed splits f, one needs to distribute the other boundaries of f into the two faces resulting from the split. The reverse effects of edge deletion should be taken care of similarly.

The halfedge data structure has also been used for representing orientable polyhedral surfaces [10]. The full power of the quadedge data structure is only realized when one deals with both subdivisions of orientable and non-orientable surfaces. To this end, one needs to introduce a *flip bit* to allow viewing the surface from the above or below. The primitives need to be enhanced for this purpose. The correctness of the data structure is proven formally using *edge algebra*. The details are in the Guibas and Stolfi's original paper [8].

The vertex split and edge contraction are also applicable for polyhedral surfaces. The edge contractibility criteria carries over straightforwardly. Edge contraction is a popular primitive for surface simplification algorithms [4, 7, 9]. The edge contractibility criteria for non-manifolds has also been studied [5].

17.8 Glossary

Arrangements. Given a collection of lines, we split each line into edges by inserting a vertex at every intersection on the line. The resulting PSLG is called the *arrangement of lines*. The *arrangement of line segments* is similarly defined.

Voronoi diagram. Let S be a set of points in the plane. For each point $p \in S$, the Voronoi region of p is defined to be $\{x \in R^2 : \|p - x\| \leq \|q - x\|, \forall q \in S\}$. The *Voronoi diagram* of S is the collection of all Voronoi regions (including their boundaries).

Triangulation. Let S be a set of points in the plane. Any maximal PSLG with the points in S as vertices is a *triangulation* of S.

Delaunay triangulation. Let S be a set of points in the plane. For any three points p, q, and r in S, if the circumcircle of the triangle pqr does not strictly enclose any point in S, we call pqr a *Delaunay triangle*. The *Delaunay triangulation* of S is the collection of all Delaunay triangles (including their boundaries). The Delaunay triangulation of S is the dual of the Voronoi diagram of S.

Acknowledgment

This work was supported, in part, by the Research Grant Council, Hong Kong, China (HKUST 6190/02E).

References

[1] B.G. Baumgart, A polyhedron representation for computer vision, *National Computer conference*, 589–596, Anaheim, CA, 1975, AFIPS.

[2] M.S. Bazaraa, J.J. Jarvis, and H.D. Sherali, *Linear Programming and Network Flows*, Wiley, 1990.

[3] M. deBerg, M. van Kreveld, M. Overmars, and O. Schwarzkopf, *Computational Geometry – Algorithms and Applications*, Springer, 2000.

[4] S.-W. Cheng, T. K. Dey, and S.-H. Poon, Hierarchy of Surface Models and Irreducible

Triangulation, *Computational Geometry: Theory and Applications*, 27(2004), 135–150.

[5] T.K. Dey, H. Edelsbrunner, S. Guha, and D.V. Nekhayev, Topology preserving edge contraction, *Publ. Inst. Math. (Beograd) (N.S.)*, 66 (1999), 23–45.

[6] C.A. Duncan, M.T. Goodrich, and S.G. Kobourov, Planarity-preserving clustering and embedding for large graphs, *Proc. Graph Drawing*, Lecture Notes Comput. Sci., Springer-Verlag, vol. 1731, 1999, 186–196.

[7] M. Garland and P.S. Heckbert. Surface simplification using quadric error metrics. *Proc. SIGGRAPH '97*, 209–216.

[8] L. Guibas and J. Stolfi, Primitives for the manipulation of general subdivisions and the computation of Voronoi diagrams, *ACM Transactions on Graphics*, 4 (1985), 74–123.

[9] H. Hoppe, T. DeRose, T. Duchamp, J. McDonald and W. Stuetzle, Mesh optimization, *Proc. SIGGRAPH '93*, 19–26.

[10] L. Kettner, Using generic programming for designing a data structure for polyhedral surfaces, *Computational Geometry - Theory and Applications*, 13 (1999), 65–90.

[11] E. Mücke, I. Saias, and B. Zhu, Fast randomized point location without preprocessing in two and three-dimensional Delaunay triangulations, *Computational Geometry: Theory and Applications*, 12(1999), 63-83, 1999.

[12] D.E. Muller and F.P. Preparata, Finding the intersection of two convex polyhedra, *Theoretical Computer Science*, 7 (1978), 217–236.

[13] F.P. Preparata and M.I. Shamos, *Computational Geometry: An Introduction*, Springer-Verlag, New York, 1985.

[14] S. Sahni, *Data Structures, Algorithms, and Applications in Java*, McGraw Hill, NY, 2000.

[15] K. Weiler, Edge-based data structures for solid modeling in curved-surface environments. *IEEE Computer Graphics and Application*, 5 (1985), 21–40.